Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Answer

The given function is $f(x) = 5x - 3$

At $x = 0$, $f(0) = 5 \times 0 - 3 = 3$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (5x - 3) = 5 \times 0 - 3 = -3$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

Therefore, *f* is continuous at *x* = 0

At $x = -3$, $f(-3) = 5 \times (-3) - 3 = -18$

$\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} (5x - 3) = 5 \times (-3) - 3 = -18$

$\therefore \lim\limits_{x \to -3} f(x) = f(-3)$

Therefore, *f* is continuous at *x* = −3

At $x = 5$, $f(x) = f(5) = 5 \times 5 - 3 = 25 - 3 = 22$

$\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5} (5x - 3) = 5 \times 5 - 3 = 22$

$\therefore \lim\limits_{x \to 5} f(x) = f(5)$

Therefore, *f* is continuous at *x* = 5

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Answer

The given function is $f(x) = 2x^2 - 1$

At $x = 3$, $f(x) = f(3) = 2 \times 3^2 - 1 = 17$

$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (2x^2 - 1) = 2 \times 3^2 - 1 = 17$

$\therefore \lim\limits_{x \to 3} f(x) = f(3)$

Thus, *f* is continuous at *x* = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x \neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x \neq -5$ (d) $f(x)=|x-5|$

Answer

(a) The given function is $f(x)=x-5$

It is evident that *f* is defined at every real number *k* and its value at *k* is *k* − 5.

It is also observed that, $\lim_{x \to k} f(x)=\lim_{x \to k}(x-5)=k-5=f(k)$

$\therefore \lim_{x \to k} f(x)=f(k)$

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x \neq 5$

For any real number *k* ≠ 5, we obtain

$$\lim_{x \to k} f(x)=\lim_{x \to k}\frac{1}{x-5}=\frac{1}{k-5}$$

$$\text{Also, } f(k)=\frac{1}{k-5} \qquad (\text{As } k \neq 5)$$

$$\therefore \lim_{x \to k} f(x)=f(k)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x \neq -5$

For any real number *c* ≠ −5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-25}{x+5} = \lim_{x\to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x<5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let *c* be a point on a real line. Then, $c < 5$ or $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : $c = 5$

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at $x = 5$

Case III: $c > 5$

$$\text{Then, } f(c) = f(5) = c-5$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Answer

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

$$\text{Then, } \lim_{x \to n} f(n) = \lim_{x \to n} \left(x^n\right) = n^n$$

$$\therefore \lim_{x \to n} f(x) = f(n)$$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer

The given function f is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

$$\text{Then, } \lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(5)=5$$

$$\therefore \lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$$

Therefore, *f* is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

$$\text{Then, } \lim_{x\to 2} f(x)=\lim_{x\to 2}(5)=5$$

$$\therefore \lim_{x\to 2} f(x)=f(2)$$

Therefore, *f* is continuous at $x = 2$

Question 6:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$$

Answer

The given function *f* is $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$

It is evident that the given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

$$\text{Then, } f(c)=2c+3$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x+3)=2c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x < 2$

Case (ii) $c > 2$

$$\text{Then, } f(c) = 2c - 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Case (iii) *c* = 2

Then, the left hand limit of *f* at *x* = 2 is,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$$

The right hand limit of *f* at *x* = 2 is,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1$$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 7:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \le -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \ge 3 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c < -3, \text{ then } f(c) = -c + 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x + 3) = -c + 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x < -3$

Case II:

$$\text{If } c = -3, \text{ then } f(-3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x+3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, *f* is continuous at $x = -3$

Case III:

$$\text{If } -3 < c < 3, \text{ then } f(c) = -2c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (-2x) = -2c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of *f* at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$$

The right hand limit of *f* at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x+2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of *f* at $x = 3$ do not coincide.

Therefore, *f* is not continuous at $x = 3$

Case V:

$$\text{If } c > 3, \text{ then } f(c) = 6c + 2 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (6x+2) = 6c + 2$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of *f*.

Question 8:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

It is known that, $x<0 \Rightarrow |x|=-x \text{ and } x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 & \text{if } x<0\\ 0, & \text{if } x=0\\ \frac{|x|}{x}=\frac{x}{x}=1, & \text{if } x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<0, \text{ then } f(c)=-1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 0

Hence, *x* = 0 is the only point of discontinuity of *f*.

Question 9:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \dfrac{x}{|x|} = \dfrac{x}{-x} = -1, \text{ if } x < 0 \\ -1, \text{ if } x \geq 0 \end{cases}$$

$$\Rightarrow f(x) = -1 \text{ for all } x \in \mathbf{R}$$

Let *c* be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases} x+1, & \text{if } x\geq 1 \\ x^2+1, & \text{if } x<1 \end{cases}$$

Answer

The given function f is $f(x)=\begin{cases} x+1, & \text{if } x\geq 1 \\ x^2+1, & \text{if } x<1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x+1)=1+1=2$

$\therefore \lim\limits_{x\to 1} f(x)=f(1)$

Therefore, f is continuous at $x = 1$

Case III:

If $c>1$, then $f(c)=c+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+1)=c+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}x^3-3, & \text{if } x\le 2\\ x^2+1, & \text{if } x>2\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<2, \text{ then } f(c)=c^3-3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(x^3-3\right)=c^3-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x < 2$

Case II:

$$\text{If } c=2, \text{ then } f(c)=f(2)=2^3-3=5$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\left(x^3-3\right)=2^3-3=5$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\left(x^2+1\right)=2^2+1=5$$

$$\therefore \lim_{x\to 2} f(x)=f(2)$$

Therefore, f is continuous at $x = 2$

Case III:

$$\text{If } c>2, \text{ then } f(c)=c^2+1$$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x > 2$

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$$

Answer

The given function f is $$f(x)=\begin{cases}x^{10}-1, & \text{if } x\le 1\\ x^2, & \text{if } x>1\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^{10}-1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$

The right hand limit of f at $x = 1$ is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}\left(x^{2}\right)=1^2=1$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{2}\right)=c^2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Question 13:

Is the function defined by

$$f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$$

a continuous function?

Answer

The given function is $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$$\text{If } c<1,\ \text{then } f(c)=c+5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

$$\text{If } c=1,\ \text{then } f(1)=1+5=6$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=1+5=6$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=1-5=-4$$

It is observed that the left and right hand limit of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

$$\text{If } c>1,\ \text{then } f(c)=c-5 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observation, it can be concluded that *x* = 1 is the only point of discontinuity of *f*.

Question 14:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function is defined at all points of the interval [0, 10].

Let *c* be a point in the interval [0, 10].

Case I:

If $0\le c<1$, then $f(c)=3$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(3)=3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous in the interval [0, 1).

Case II:

If $c=1$, then $f(3)=3$

The left hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(3)=3$

The right hand limit of *f* at *x* = 1 is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(4)=4$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case III:

If $1<c<3$, then $f(c)=4$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(4)=4$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval (1, 3).

Case IV:

If $c=3$, then $f(c)=5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(5) = 5$$

$$\lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ and $x = 3$

Question 15:

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c < 0, \text{ then } f(c) = 2c$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0$$

The left hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of *f* at *x* = 0 is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

Case III:

$$\text{If } 0 < c < 1, \text{ then } f(x) = 0 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 0$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of *f* at *x* = 1 do not coincide.

Therefore, *f* is not continuous at *x* = 1

Case V:

$$\text{If } c < 1, \text{ then } f(c) = 4c \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Hence, *f* is not continuous only at *x* = 1

Question 16:

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x < -1$

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of f at $x = -1$ is,

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(-2)=-2$

The right hand limit of f at $x = -1$ is,

$\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}(2x)=2\times(-1)=-2$

$\therefore \lim\limits_{x\to -1} f(x)=f(-1)$

Therefore, f is continuous at $x = -1$

Case III:

If $-1<c<1$, then $f(c)=2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points of the interval $(-1, 1)$.

Case IV:

$$\text{If } c=1, \text{ then } f(c)=f(1)=2\times 1=2$$

The left hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(2x)=2\times 1=2$$

The right hand limit of *f* at *x* = 1 is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2=2$$

$$\therefore \lim_{x\to 1} f(x)=f(c)$$

Therefore, *f* is continuous at *x* = 2

Case V:

$$\text{If } c>1, \text{ then } f(c)=2 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(2)=2$$

$$\lim_{x\to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 1

Thus, from the above observations, it can be concluded that *f* is continuous at all points of the real line.

Question 17:

Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$$

is continuous at *x* = 3.

Answer

The given function *f* is $f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$

If *f* is continuous at *x* = 3, then

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \qquad ...(1)$$

Also,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda\left(x^2-2x\right), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function f is $f(x) = \begin{cases} \lambda\left(x^2-2x\right), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$

If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \lambda\left(x^2-2x\right) = \lim_{x\to 0^+} (4x+1) = \lambda\left(0^2-2\times 0\right)$$

$$\Rightarrow \lambda\left(0^2-2\times 0\right) = 4\times 0+1 = 0$$

$$\Rightarrow 0 = 1 = 0, \text{ which is not possible}$$

Therefore, there is no value of λ for which f is continuous at x = 0

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x \to 1} (4x+1) = 4 \times 1 + 1 = 5$$

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

Therefore, for any values of λ, f is continuous at $x = 1$

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - [x]) = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x] = n - (n-1) = 1$$

The right hand limit of f at $x = n$ is,

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - [x]) = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = p?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that *f* is defined at *x* = p

At $x=\pi$, $f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$

Consider $\lim\limits_{x\to\pi} f(x)=\lim\limits_{x\to\pi}\left(x^2-\sin x+5\right)$

Put $x=\pi+h$

If $x\to\pi$, *then it is evident that* $h\to 0$

$$\begin{aligned}\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2-\sin x+5\right)\\ &= \lim_{h\to 0}\left[(\pi+h)^2-\sin(\pi+h)+5\right]\\ &= \lim_{h\to 0}(\pi+h)^2-\lim_{h\to 0}\sin(\pi+h)+\lim_{h\to 0}5\\ &= (\pi+0)^2-\lim_{h\to 0}\left[\sin\pi\cosh+\cos\pi\sinh\right]+5\\ &= \pi^2-\lim_{h\to 0}\sin\pi\cosh-\lim_{h\to 0}\cos\pi\sinh+5\\ &= \pi^2-\sin\pi\cos 0-\cos\pi\sin 0+5\\ &= \pi^2-0\times 1-(-1)\times 0+5\\ &= \pi^2+5\end{aligned}$$

$$\therefore \lim_{x\to\pi} f(x)=f(\pi)$$

Therefore, the given function *f* is continuous at *x* = π

Question 21:

Discuss the continuity of the following functions.

(a) *f* (*x*) = sin *x* + cos *x*

(b) *f* (*x*) = sin *x* − cos *x*

(c) *f* (*x*) = sin *x* × cos x

Answer

It is known that if *g* and *h* are two continuous functions, then

$g+h,\ g-h,\ \text{and } g.h$ are also continuous.

It has to proved first that *g* (*x*) = sin *x* and *h* (*x*) = cos *x* are continuous functions.

Let *g* (*x*) = sin *x*

It is evident that *g* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

$$g(c) = \sin c$$

$$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$$
$$= \lim_{h \to 0} \sin(c+h)$$
$$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$$
$$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let *h* (*x*) = cos *x*

It is evident that *h* (*x*) = cos *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *h*

If *x* → *c*, then *h* → 0

h (*c*) = cos *c*

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos(c+h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, *h* is a continuous function.

Therefore, it can be concluded that

(a) *f* (*x*) = *g* (*x*) + *h* (*x*) = sin *x* + cos *x* is a continuous function

(b) *f* (*x*) = *g* (*x*) − *h* (*x*) = sin *x* − cos *x* is a continuous function

(c) *f* (*x*) = *g* (*x*) × *h* (*x*) = sin *x* × cos *x* is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if *g* and *h* are two continuous functions, then

$(i)\ \dfrac{h(x)}{g(x)},\ g(x) \neq 0$ is continuous

$(ii)\ \dfrac{1}{g(x)},\ g(x) \neq 0$ is continuous

$(iii)\ \dfrac{1}{h(x)},\ h(x) \neq 0$ is continuous

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) = \lim_{x\to c} \sin x \\ &= \lim_{h\to 0} \sin(c+h) \\ &= \lim_{h\to 0}\left[\sin c \cos h + \cos c \sin h\right] \\ &= \lim_{h\to 0}(\sin c \cos h) + \lim_{h\to 0}(\cos c \sin h) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$

$$\therefore\ \lim_{x\to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If *x* ® *c*, then *h* ® 0

$h(c) = \cos c$

$$\lim_{x\to c} h(x) = \lim_{x\to c} \cos x$$
$$= \lim_{h\to 0} \cos(c+h)$$
$$= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h\to 0} \cos c \cos h - \lim_{h\to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *h* (*x*) = cos *x* is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at *x* = *n*p, *n* Î **Z**

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at *x* = *n*p, *n* Î **Z**

Question 23:

Find the points of discontinuity of *f*, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, \text{ if } x < 0 \\ x+1, \text{ if } x \geq 0 \end{cases}$$

Answer

The given function f is $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\geq 0\end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x > 0$

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous function?

Answer

The given function f is $$f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c)=c^2 \sin\frac{1}{c}$$

$$\lim_{x \to c} f(x)=\lim_{x \to c}\left(x^2 \sin\frac{1}{x}\right)=\left(\lim_{x \to c} x^2\right)\left(\lim_{x \to c} \sin\frac{1}{x}\right)=c^2 \sin\frac{1}{c}$$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

$$\text{If } c=0, \text{ then } f(0)=0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 25:

Examine the continuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

$$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$$
$$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* ≠ 0

Case II:

$$\text{If } c = 0, \text{ then } f(0) = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, *f* is continuous at *x* = 0

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function.

Question 26:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer

The given function *f* is $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function *f* is continuous at $x = \frac{\pi}{2}$, if *f* is defined at $x = \frac{\pi}{2}$ and if the value of the *f* at $x = \frac{\pi}{2}$ equals the limit of *f* at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

$$\text{Put } x=\frac{\pi}{2}+h$$

$$\text{Then, } x\to\frac{\pi}{2}\Rightarrow h\to 0$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases} \qquad \text{at } x=2$$

Answer

The given function is $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$

The given function f is continuous at x = 2, if f is defined at x = 2 and if the value of f at x = 2 equals the limit of f at x = 2

It is evident that f is defined at x = 2 and $f(2)=k(2)^2=4k$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \to 2^-} \left(kx^2\right) = \lim_{x \to 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function *f* is continuous at *x* = p, if *f* is defined at *x* = p and if the value of *f* at *x* = p equals the limit of *f* at *x* = p

It is evident that *f* is defined at *x* = p and $f(\pi) = k\pi + 1$

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \to \pi^-} (kx + 1) = \lim_{x \to \pi^+} \cos x = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases}kx+1, & \text{if } x\le 5\\ 3x-5, & \text{if } x>5\end{cases} \qquad \text{at } x=5$$

Answer

The given function *f* is $f(x)=\begin{cases}kx+1, & \text{if } x\le 5\\ 3x-5, & \text{if } x>5\end{cases}$

The given function *f* is continuous at *x* = 5, if *f* is defined at *x* = 5 and if the value of *f* at *x* = 5 equals the limit of *f* at *x* = 5

It is evident that *f* is defined at *x* = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$

$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$$\Rightarrow 5k=9$$

$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of *a* and *b* such that the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

is a continuous function.

Answer

The given function f is $f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10

Since f is continuous at x = 2, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad ...(1)$$

Since f is continuous at x = 10, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad ...(2)$$

On subtracting equation (1) from equation (2), we obtain

8a = 16

⇒ a = 2

By putting a = 2 in equation (1), we obtain

2 × 2 + b = 5

⇒ 4 + b = 5

$\Rightarrow b = 1$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer

The given function is $f(x) = \cos(x^2)$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that *g* is defined for every real number.

Let *c* be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}\lim_{x \to c} g(x) &= \lim_{x \to c} \cos x \\ &= \lim_{h \to 0} \cos(c+h) \\ &= \lim_{h \to 0} \left[\cos c \cos h - \sin c \sin h\right] \\ &= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c \times 1 - \sin c \times 0 \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, *h* is defined for every real number.

Let *k* be a real number, then $h(k) = k^2$

$$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$$

$$\therefore \lim_{x \to k} h(x) = h(k)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer

The given function is $f(x) = |\cos x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g* o *h*, where $g(x) = |x|$ and $h(x) = \cos x$

$$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

$$\text{If } c > 0, \text{ then } g(c) = c \text{ and } \lim_{x \to c} g(x) = \lim_{x \to c} x = c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

$$\text{If } c = 0, \text{ then } g(c) = g(0) = 0$$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$

$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c \times 1 - \sin c \times 0 \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h,such that $(g \text{ o } h)$ is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then $(f \text{ o } g)$ is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x)=\sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

f = *g o h*, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0 \\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim_{x\to c} g(x)=\lim_{x\to c} x=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* > 0

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+}(x) = g(0)$$

Therefore, *g* is continuous at *x* = 0

From the above three observations, it can be concluded that *g* is continuous at all points.

h (*x*) = sin *x*

It is evident that *h* (*x*) = sin *x* is defined for every real number.

Let *c* be a real number. Put *x* = *c* + *k*

If *x* → *c*, then *k* → 0

h (*c*) = sin *c*

$$h(c) = \sin c$$
$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \sin x \\ &= \lim_{k\to 0} \sin(c+k) \\ &= \lim_{k\to 0}[\sin c \cos k + \cos c \sin k] \\ &= \lim_{k\to 0}(\sin c \cos k) + \lim_{h\to 0}(\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$
$$\therefore \lim_{x\to c} h(x) = g(c)$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of *f* defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, *g* and *h*, are defined as

$$g(x)=|x| \text{ and } h(x)=|x+1|$$

Then, $f = g - h$

The continuity of *g* and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases} -x, & \text{if } x<0 \\ x, & \text{if } x\geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x < 0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x > 0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$$

Therefore, *g* is continuous at $x = 0$

From the above three observations, it can be concluded that *g* is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases} -(x+1), & \text{if, } x<-1 \\ x+1, & \text{if } x\geq -1 \end{cases}$$

Clearly, *h* is defined for every real number.

Let c be a real number.

Case I:

$$\text{If } c < -1, \text{ then } h(c) = -(c+1) \text{ and } \lim_{x \to c} h(x) = \lim_{x \to c}\left[-(x+1)\right] = -(c+1)$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

$$\text{If } c > -1, \text{ then } h(c) = c+1 \text{ and } \lim_{x \to c} h(x) = \lim_{x \to c}(x+1) = c+1$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

$$\text{If } c = -1, \text{ then } h(c) = h(-1) = -1+1 = 0$$

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-}\left[-(x+1)\right] = -(-1+1) = 0$$

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+}(x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \to -1^-} h(x) = \lim_{h \to -1^+} h(x) = h(-1)$$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to *x*.

$\sin\left(x^2+5\right)$

Answer

$\text{Let } f(x)=\sin\left(x^2+5\right),\ u(x)=x^2+5, \text{ and } v(t)=\sin t$

$\text{Then, } (vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, *f* is a composite of two functions.

$\text{Put } t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

$$\text{Therefore, by chain rule, } \frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to *x*.

$\cos(\sin x)$

Answer

$$\text{Let } f(x)=\cos(\sin x), u(x)=\sin x, \text{ and } v(t)=\cos t$$
$$\text{Then, } (vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$$

Thus, *f* is a composite function of two functions.

Put *t* = *u* (*x*) = sin *x*

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$
$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$$

Question 3:

Differentiate the functions with respect to *x*.

$$\sin(ax+b)$$

Answer

$$\text{Let } f(x)=\sin(ax+b), \ u(x)=ax+b, \text{ and } v(t)=\sin t$$
$$\text{Then, } (vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put *t* = *u* (*x*) = *ax* + *b*

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$
$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

Question 4:

Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer

$$\text{Let } f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t, \text{and } w(s)=\sec s$$

$$\text{Then, } (wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$$

Thus, *f* is a composite function of three functions, *u, v*, and *w*.

$$\text{Put } s=v(t)=\tan t \text{ and } t=u(x)=\sqrt{x}$$

$$\text{Then, } \frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t) \qquad [s=\tan t]$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) \qquad \left[t=\sqrt{x}\right]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to *x*.

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x)=\sin(ax+b)$ and $h(x)=\cos(cx+d)$

$$\therefore f'=\frac{g'h-gh'}{h^2}$$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ *g* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$$

Consider $h(x) = \cos(cx+d)$

Let $p(x) = cx + d,\ q(y) = \cos y$

Then, $(qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$

$\therefore$ *h* is a composite function of two functions, *p* and *q*.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx+d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx+d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain

$$h' = \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} = -\sin(cx+d) \times c = -c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to *x*.

$\cos x^3.\sin^2(x^5)$

Answer

The given function is $\cos x^3.\sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3) + \cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3) + \cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2(x^5)\times 3x^2 + 2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$

$$= -3x^2\sin x^3\cdot\sin^2(x^5) + 2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3 - 3x^2\sin x^3\sin^2(x^5)$$

Question 7:

Differentiate the functions with respect to *x*.

$2\sqrt{\cot(x^2)}$

Answer

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]$$

$$=2\cdot\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times-\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$=-\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times(2x)$$

$$=\frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\sin x^2}$$

$$=\frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the functions with respect to *x*.

$$\cos\left(\sqrt{x}\right)$$

Answer

$$\text{Let } f(x)=\cos\left(\sqrt{x}\right)$$

$$\text{Also, let } u(x)=\sqrt{x}$$

$$\text{And, } v(t)=\cos t$$

$$\begin{aligned}\text{Then, } (vou)(x)&=v\left(u(x)\right)\\&=v\left(\sqrt{x}\right)\\&=\cos\sqrt{x}\\&=f(x)\end{aligned}$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$$t=u(x)=\sqrt{x}$$

Then, $\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}$

$$=\frac{1}{2\sqrt{x}}$$

And, $\frac{dv}{dt}=\frac{d}{dt}\left(\cos t\right)=-\sin t$

$$=-\sin\left(\sqrt{x}\right)$$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}$$

$$=-\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Question 9:

Prove that the function *f* given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x)=|x-1|,\ x\in\mathbf{R}$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \text{ and } \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$ are finite and equal.

To check the differentiability of the given function at *x* = 1,

consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^-}\frac{|h|-0}{h} = \lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0 \Rightarrow |h| = -h)$$

$$= -1$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^+}\frac{|h|-0}{h} = \lim_{h\to 0^+}\frac{h}{h} \qquad (h>0 \Rightarrow |h| = h)$$

$$= 1$$

Since the left and right hand limits of *f* at *x* = 1 are not equal, *f* is not differentiable at *x* = 1

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at *x* = 1 and *x* = 2.

Answer

The given function *f* is $f(x) = [x], 0 < x < 3$

It is known that a function *f* is differentiable at a point *x* = *c* in its domain if both

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \text{ and } \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$ are finite and equal.

To check the differentiability of the given function at *x* = 1, consider the left hand limit of *f* at *x* = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^-}\frac{1-2}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$

$$=\lim_{h\to 0^+}\frac{2-2}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2

Question 1:

Find $\frac{dy}{dx}$:

$2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$
$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$
$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$
$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$:

$2x+3y=\sin y$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$:

$ax+by^2=\cos y$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ ...(2)

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$:

$xy+y^2=\tan x+y$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \left[\text{Using product rule and chain rule}\right]$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}$:

$x^2 + xy + y^2 = 100$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^2 + xy + y^2\right) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$ [Derivative of constant function is 0]

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Question 6:

Find $\frac{dy}{dx}$:

$x^3+x^2y+xy^2+y^3=81$

Answer

The given relationship is $x^3+x^2y+xy^2+y^3=81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(x^3+x^2y+xy^2+y^3\right)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy^2\right)+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\cdot 2x+x^2\frac{dy}{dx}\right]+\left[y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dy}{dx}$:

$\sin^2 y+\cos xy=\pi$

Answer

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}(\cos xy) = 0 \qquad \text{...(1)}$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad \text{...(2)}$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad \text{...(3)}$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\frac{dy}{dx} = 0$$

$$\Rightarrow \left(2\sin y\cos y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \left(\sin 2y - x\sin xy\right)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$:

$\sin^2 x + \cos^2 y = 1$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x\cdot\frac{d}{dx}(\sin x)+2\cos y\cdot\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x\cdot[0+2x]}{\left(1+x^2\right)^2}=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$:

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2} \qquad ...(1)$$

It is known that,

$$\tan y=\frac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}} \qquad ...(2)$$

Comparing equations (1) and (2), we obtain

$$x=\tan\frac{y}{3}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

Question 11:

Find $\frac{dy}{dx}$:

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$$

x

Answer

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to *x*, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \ldots(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y\cdot\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right)\cdot\left(1-x^2\right)'-\left(1-x^2\right)\cdot\left(1+x^2\right)'}{\left(1+x^2\right)^2} \qquad \left[\text{Using quotient rule}\right]$$

$$=\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)\cdot(2x)}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx} \cdot \left[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right] \cdot \left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1 = \left(1 + x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1 + x^2}$$

Question 13:

Find $\frac{dy}{dx}$:

$$y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right), -1 < x < 1$$

Answer

The given relationship is $y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$

$$y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow \cos y = \frac{2x}{1 + x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y) = \frac{d}{dx} \cdot \left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{\left(1 + x^2\right) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}\left(1 + x^2\right)}{\left(1 + x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx} = \frac{\left(1+x^2\right)\times 2 - 2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2 - 4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}$:

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Answer

lationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\cos y \frac{dy}{dx} = 2\left[x \frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$:

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$$

Answer

The given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x \cdot (\cos x)}{\sin^2 x}$$

$$= \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Question 2:

Differentiate the following w.r.t. x:

$$e^{\sin^{-1} x}$$

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow\frac{dy}{dx}=e^{\sin^{-1}x}\cdot\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore\frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},x\in(-1,1)$$

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer

Let x_1 and x_2 be any two numbers in **R**.

Then, we have:

$$x_1<x_2\Rightarrow 2x_1<2x_2\Rightarrow e^{2x_1}<e^{2x_2}\Rightarrow f\left(x_1\right)<f\left(x_2\right)$$

Hence, *f* is strictly increasing on **R**.

Question 3:

Differentiate the following w.r.t. x:

e^{x^3}

Answer

Let $y=e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x^3}\right)=e^{x^3}\cdot\frac{d}{dx}\left(x^3\right)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$$

Question 4:

Differentiate the following w.r.t. x:

$\sin\left(\tan^{-1}e^{-x}\right)$

Answer

Let $y=\sin\left(\tan^{-1}e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(\tan^{-1}e^{-x}\right)\right]$$
$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)$$
$$=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^{2}}\cdot\frac{d}{dx}\left(e^{-x}\right)$$
$$=\frac{\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x)$$
$$=\frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\times(-1)$$
$$=\frac{-e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following w.r.t. *x*:

$\log\left(\cos e^{x}\right)$

Answer

Let $y=\log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^{x}\right)\right]$$
$$=\frac{1}{\cos e^{x}}\cdot\frac{d}{dx}\left(\cos e^{x}\right)$$
$$=\frac{1}{\cos e^{x}}\cdot\left(-\sin e^{x}\right)\cdot\frac{d}{dx}\left(e^{x}\right)$$
$$=\frac{-\sin e^{x}}{\cos e^{x}}\cdot e^{x}$$
$$=-e^{x}\tan e^{x},\ e^{x}\neq(2n+1)\frac{\pi}{2},\ n\in\mathbf{N}$$

Question 6:

Differentiate the following w.r.t. x:

$e^x + e^{x^2} + \ldots + e^{x^5}$

Answer

$$\frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$

$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$

$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$

$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Question 7:

Differentiate the following w.r.t. x:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},\ x > 0$$

Question 8:

Differentiate the following w.r.t. *x*:

$\log(\log x), x > 1$

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x}\cdot\frac{1}{x}$$

$= \dfrac{1}{x\log x}$, $x > 1$

Question 9:

Differentiate the following w.r.t. *x*:

$\dfrac{\cos x}{\log x}, x > 0$

Answer

Let $y = \dfrac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x \log x . \sin x + \cos x]}{x(\log x)^2}, x > 0$$

Question 10:

Differentiate the following w.r.t. *x*:

$\cos\left(\log x + e^x\right), x > 0$

Answer

Let $y = \cos\left(\log x + e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = -\sin\left(\log x + e^x\right) \cdot \frac{d}{dx}\left(\log x + e^x\right)$$

$$= -\sin\left(\log x + e^x\right) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}\left(e^x\right)\right]$$

$$= -\sin\left(\log x + e^x\right) \cdot \left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin\left(\log x + e^x\right), x > 0$$

Question 1:

Differentiate the function with respect to *x*.

$\cos x.\cos 2x.\cos 3x$

Answer

Let $y=\cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left(\cos x.\cos 2x.\cos 3x\right)$$
$$\Rightarrow \log y=\log\left(\cos x\right)+\log\left(\cos 2x\right)+\log\left(\cos 3x\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}\cdot\frac{d}{dx}\left(\cos x\right)+\frac{1}{\cos 2x}\cdot\frac{d}{dx}\left(\cos 2x\right)+\frac{1}{\cos 3x}\cdot\frac{d}{dx}\left(\cos 3x\right)$$
$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}\left(2x\right)-\frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}\left(3x\right)\right]$$
$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x\left[\tan x+2\tan 2x+3\tan 3x\right]$$

Question 2:

Differentiate the function with respect to *x*.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer

$$\text{Let } y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)+\frac{1}{x-2}\cdot\frac{d}{dx}(x-2)-\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)\\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4)-\frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to *x*.

$$(\log x)^{\cos x}$$

Answer

$$\text{Let } y=(\log x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\log y=\cos x\cdot\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to *x*.

$x^x-2^{\sin x}$

Answer

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u=x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log x+x\times\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x(\log x+1)$$

$$\Rightarrow\frac{du}{dx}=x^x(1+\log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to *x*.

$$(x+3)^2 . (x+4)^3 . (x+5)^4$$

Answer

$$\text{Let } y = (x+3)^2 . (x+4)^3 . (x+5)^4$$

Taking logarithm on both the sides, we obtain

$$\log y = \log (x+3)^2 + \log (x+4)^3 + \log (x+5)^4$$

$$\Rightarrow \log y = 2 \log (x+3) + 3 \log (x+4) + 4 \log (x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y} . \frac{dy}{dx} = 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 \cdot \left[2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12) \right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2 (x+5)^3 (9x^2 + 70x + 133)$$

Question 6:

Differentiate the function with respect to *x*.

$$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

Answer

$$\text{Let } y=\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

$$\text{Also, let } u=\left(x+\frac{1}{x}\right)^{x} \text{ and } v=x^{\left(1+\frac{1}{x}\right)}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

$$\text{Then, } u=\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \quad ...(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)\qquad\ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

Differentiate the function with respect to *x*.

$$(\log x)^x+x^{\log x}$$

Answer

Let $y=(\log x)^x+x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\ldots(1)$$

$u = (\log x)^x$

$$\Rightarrow\log u=\log\left[(\log x)^x\right]$$

$$\Rightarrow\log u=x\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad\ldots(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad\ldots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Question 8:

Differentiate the function with respect to *x*.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \cdot \log(\sin x) + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x (x\cot x + \log\sin x) \qquad ...(2)$$

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x (x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to *x*.

$$x^{\sin x}+(\sin x)^{\cos x}$$

Answer

$$\text{Let } y=x^{\sin x}+(\sin x)^{\cos x}$$

$$\text{Also, let } u=x^{\sin x} \text{ and } v=(\sin x)^{\cos x}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

$$u=x^{\sin x}$$

$$\Rightarrow \log u=\log\left(x^{\sin x}\right)$$

$$\Rightarrow \log u=\sin x\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad \ldots(2)$$

$$v=(\sin x)^{\cos x}$$

$$\Rightarrow \log v=\log(\sin x)^{\cos x}$$

$$\Rightarrow \log v=\cos x\log(\sin x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\cot x\cos x\right]$$

$$\Rightarrow\frac{dv}{dx}=(\sin x)^{\cos x}\left[\cot x\cos x-\sin x\log\sin x\right] \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

Question 10:

Differentiate the function with respect to *x*.

$$x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

Answer

$$\text{Let } y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$$

$$\text{Also, let } u=x^{x\cos x} \text{ and } v=\frac{x^2+1}{x^2-1}$$

$$\therefore y=u+v$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad ...(1)$$

$$u=x^{x\cos x}$$

$$\Rightarrow\log u=\log\left(x^{x\cos x}\right)$$

$$\Rightarrow\log u=x\cos x\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}(\cos x\log x-x\sin x\log x+\cos x)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right] \quad ...(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to *x*.

$$(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$$

Answer

$$\text{Let } y=(x\cos x)^{x}+(x\sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u=(x\cos x)^{x} \text{ and } v=(x\sin x)^{\frac{1}{x}}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$$u=(x\cos x)^{x}$$

$$\Rightarrow \log u=\log(x\cos x)^{x}$$

$$\Rightarrow \log u=x\log(x\cos x)$$

$$\Rightarrow \log u=x[\log x+\log\cos x]$$

$$\Rightarrow \log u=x\log x+x\log\cos x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[\left(\log x\cdot 1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot 1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[(\log x+1)+\left\{\log\cos x+\frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[(1+\log x)+(\log\cos x-x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[1-x\tan x+(\log x+\log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}\left[1-x\tan x+\log(x\cos x)\right] \qquad ...(2)$$

$$v=(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{x}\right]+\left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[-\frac{\log(\sin x)}{x^2}+\frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right] \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=(x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad ...(1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad ...(2)$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$y^x=x^y$

Answer

The given function is $y^x=x^y$

Taking logarithm on both the sides, we obtain

$$x\log y=y\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

$(\cos x)^y = (\cos y)^x$

Answer

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$xy = e^{(x-y)}$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Answer

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

$$\text{Hence, } f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times 1}{1+1^2}+\frac{4\times 1^3}{1+1^4}+\frac{8\times 1^7}{1+1^8}\right]$$

$$=2\times 2\times 2\times 2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Question 17:

Differentiate $(x^5-5x+8)(x^3+7x+9)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer

Let $y=(x^5-5x+8)(x^3+7x+9)$

(i)

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$\therefore y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(x^2-5x+8\right)\cdot\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)\left(x^3+7x+9\right)+\left(x^2-5x+8\right)\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+x^2\left(3x^2+7\right)-5x\left(3x^2+7\right)+8\left(3x^2+7\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4+7x^2\right)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii)

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$
$$=x^2\left(x^3+7x+9\right)-5x\left(x^3+7x+9\right)+8\left(x^3+7x+9\right)$$
$$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$
$$=x^5-5x^4+15x^3-26x^2+11x+72$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(x^5-5x^4+15x^3-26x^2+11x+72\right)$$
$$=\frac{d}{dx}\left(x^5\right)-5\frac{d}{dx}\left(x^4\right)+15\frac{d}{dx}\left(x^3\right)-26\frac{d}{dx}\left(x^2\right)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$
$$=5x^4-5\times4x^3+15\times3x^2-26\times2x+11\times1+0$$
$$=5x^4-20x^3+45x^2-52x+11$$

(iii) $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If *u*, *v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y=u.v.w=u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v\cdot w)$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v\cdot w+u\left[\frac{dv}{dx}\cdot w+v\cdot\frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w$, we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=u.v.w.\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

Question 1:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2,\ y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \cos \theta,\ y = b \cos \theta$

Answer

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t$, $y = \cos 2t$

Answer

The given equations are $x = \sin t$ and $y = \cos 2t$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = 4t,\ y = \frac{4}{t}$$

Answer

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)=4\cdot\frac{d}{dt}\left(\frac{1}{t}\right)=4\cdot\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}$$

Question 5:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Answer

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

$$\text{Then, } \frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$$
$$=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$$
$$=\cos\theta-2\cos 2\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

Question 6:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, \; y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t)-\cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t \sin 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t.\sin t + \cos^3 t(2\sin t\cos t)}{3\cos 2t \sin^2 t \cos t + \sin^3 t(2\sin t \cos t)}$$

$$= \frac{\sin t\cos t\left[-3\cos 2t.\cos t + 2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t + 2\sin^3 t\right]}$$

$$= \frac{\left[-3\left(2\cos^2 t - 1\right)\cos t + 2\cos^3 t\right]}{\left[3\left(1 - 2\sin^2 t\right)\sin t + 2\sin^3 t\right]} \qquad \begin{bmatrix} \cos 2t = \left(2\cos^2 t - 1\right), \\ \cos 2t = \left(1 - 2\sin^2 t\right) \end{bmatrix}$$

$$= \frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= \frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix} \cos 3t = 4\cos^3 t - 3\cos t, \\ \sin 3t = 3\sin t - 4\sin^3 t \end{bmatrix}$$

$$= -\cot 3t$$

Question 8:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a\sin t$

Answer

The given equations are $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a\sin t$

Then, $\frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \sec \theta,\ y = b \tan \theta$

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

$$\text{Then, } \frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\cdot\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If *x* and *y* are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}},\ y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$$x=\sqrt{a^{\sin^{-1}t}} \text{ and } y=\sqrt{a^{\cos^{-1}t}}$$

$$\Rightarrow x=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}} \text{ and } y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$$

$$\Rightarrow x=a^{\frac{1}{2}\sin^{-1}t} \text{ and } y=a^{\frac{1}{2}\cos^{-1}t}$$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log x=\frac{1}{2}\sin^{-1}t\log a$$

$$\therefore \frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)}=-\frac{y}{x}.$$

Hence, proved.

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

$x^2 + 3x + 2$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

x^{20}

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$x \cdot \cos x$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x\sin x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x\sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)$$

$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)\right]$$

$$= -\sin x - (\sin x + x\cos x)$$

$$= -(x\cos x + 2\sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3 \log x$

Answer

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x\cdot\frac{d}{dx}\left(x^3\right)+x^3\cdot\frac{d}{dx}(\log x)$$
$$=\log x\cdot 3x^2+x^3\cdot\frac{1}{x}=\log x\cdot 3x^2+x^2$$
$$=x^2\left(1+3\log x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2\left(1+3\log x\right)\right]$$
$$=\left(1+3\log x\right)\cdot\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}\left(1+3\log x\right)$$
$$=\left(1+3\log x\right)\cdot 2x+x^2\cdot\frac{3}{x}$$
$$=2x+6x\log x+3x$$
$$=5x+6x\log x$$
$$=x\left(5+6\log x\right)$$

Question 6:

Find the second order derivatives of the function.

$e^x\sin 5x$

Answer

Let $y=e^x\sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)=\sin 5x\cdot\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}\left(\sin 5x\right)$$
$$=\sin 5x\cdot e^x+e^x\cdot\cos 5x\cdot\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x\cdot 5$$
$$=e^x\left(\sin 5x+5\cos 5x\right)$$
$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x\left(\sin 5x+5\cos 5x\right)\right]$$
$$=\left(\sin 5x+5\cos 5x\right)\cdot\frac{d}{dx}\left(e^x\right)+e^x\cdot\frac{d}{dx}\left(\sin 5x+5\cos 5x\right)$$
$$=\left(\sin 5x+5\cos 5x\right)e^x+e^x\left[\cos 5x\cdot\frac{d}{dx}(5x)+5\left(-\sin 5x\right)\cdot\frac{d}{dx}(5x)\right]$$
$$=e^x\left(\sin 5x+5\cos 5x\right)+e^x\left(5\cos 5x-25\sin 5x\right)$$

Then,
$$=e^x\left(10\cos 5x-24\sin 5x\right)=2e^x\left(5\cos 5x-12\sin 5x\right)$$

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cdot\cos 3x\right) = \cos 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\cos 3x)$$

$$= \cos 3x\cdot e^{6x}\cdot\frac{d}{dx}(6x) + e^{6x}\cdot(-\sin 3x)\cdot\frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6\cdot\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3\cdot\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6\cdot\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\cdot\left[\sin 3x\cdot\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\cdot\frac{d}{dx}(\sin 3x)\right] \qquad [\text{Using } (1)]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x\cdot e^{6x}\cdot 6 + e^{6x}\cdot\cos 3x\cdot 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Question 8:

Find the second order derivatives of the function.

$\tan^{-1} x$

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$=\frac{-1}{\left(1+x^2\right)^2}\times 2x=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivatives of the function.

$\log(\log x)$

Answer

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)=\frac{1}{x\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]=\frac{-(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivatives of the function.

$\sin(\log x)$

Answer

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\log x)\right]=\cos(\log x)\cdot\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right]-\cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$=\frac{-x\sin(\log x).\frac{1}{x}-\cos(\log x)}{x^2}$$

$$=\frac{-\left[\sin(\log x)+\cos(\log x)\right]}{x^2}$$

Question 11:

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Answer

It is given that, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$
$$=5(-\sin x)-3\cos x=-(5\sin x+3\cos x)$$
$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(5\sin x+3\cos x)\right]$$
$$=-\left[5\cdot\frac{d}{dx}(\sin x)+3\cdot\frac{d}{dx}(\cos x)\right]$$
$$=-\left[5\cos x+3(-\sin x)\right]$$
$$=-\left[5\cos x-3\sin x\right]$$
$$=-y$$
$$\therefore \frac{d^2y}{dx^2}+y=0$$

Hence, proved.

Question 12:

If $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of *y* alone.

Answer

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}}\qquad\ldots\text{(i)}$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Answer

It is given that, $y=3\cos(\log x)+4\sin(\log x)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos(\log x)\right] + 4\cdot\frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] + 4\cdot\left[\cos(\log x)\cdot\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - 3\{\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}\left(e^{mx}\right) + B \cdot \frac{d}{dx}\left(e^{nx}\right) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}\left(e^{mx}\right) + Bn \cdot \frac{d}{dx}\left(e^{nx}\right)$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$=500\cdot e^{7x}\cdot\frac{d}{dx}(7x)+600\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$

$$=3500e^{7x}-4200e^{-7x}$$

$$\therefore\frac{d^2y}{dx^2}=3500\cdot\frac{d}{dx}\left(e^{7x}\right)-4200.\frac{d}{dx}\left(e^{-7x}\right)$$

$$=3500\cdot e^{7x}\cdot\frac{d}{dx}(7x)-4200\cdot e^{-7x}\cdot\frac{d}{dx}(-7x)$$

$$=7\times3500\cdot e^{7x}+7\times4200\cdot e^{-7x}$$

$$=49\times500e^{7x}+49\times600e^{-7x}$$

$$=49\left(500e^{7x}+600e^{-7x}\right)$$

$$=49y$$

Hence, proved.

Question 16:

If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1)=1$

$$e^y(x+1)=1$$

$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1=2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2y_2+2x\left(1+x^2\right)y_1=2$$

Hence, proved.

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$

Answer

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$

$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$

∴ $f(-4) = f(2) = 0$

⇒ The value of $f(x)$ at −4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$f(x) = x^2 + 2x - 8$

$\Rightarrow f'(x) = 2x + 2$

$\therefore f'(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow c = -1, \text{ where } c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

Answer

By Rolle's Theorem, for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

(c) *f* (*a*) = *f* (*b*)

then, there exists some *c* ∈ (*a*, *b*) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

$\text{Also, } f(5)=[5]=5 \text{ and } f(9)=[9]=9$

$\therefore f(5)\neq f(9)$

The differentiability of *f* in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$$f(1) = (1)^2 - 1 = 0$$

$$f(2) = (2)^2 - 1 = 3$$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=x^2-1 \text{ for } x\in[1,\ 2]$.

Question 3:

If $f:[-5,5]\to\mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Answer

It is given that $f:[-5,5]\to\mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) *f* is continuous on [−5, 5].

(b) *f* is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists *c* ∈ (−5, 5) such that

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$

$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c)\neq 0$$

$$\Rightarrow 10f'(c)\neq 0$$

$$\Rightarrow f(5)-f(-5)\neq 0$$

$$\Rightarrow f(5)\neq f(-5)$$

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a,\ b]$, where $a=1$ and $b=4$.

Answer

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is $2x - 4$.

$$f(1) = 1^2 - 4\times 1 - 3 = -6,\ f(4) = 4^2 - 4\times 4 - 3 = -3$$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1} = \frac{-3-(-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$f'(c) = 1$$
$$\Rightarrow 2c - 4 = 1$$
$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1,\ 4)$$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval [*a*, *b*], where *a* = 1 and *b* = 3. Find all $c \in (1,3)$ for which $f'(c) = 0$

Answer

The given function *f* is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5\times 1^2 - 3\times 1 = -7,\ \ f(3) = 3^3 - 5\times 3^2 - 3\times 3 = -27$$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = -10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$$f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = 10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$
$$\Rightarrow (c-1)(3c-7) = 0$$
$$\Rightarrow c = 1, \ \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, \ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1, \ 3)$ is the only point for which $f'(c) = 0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f : [a, \text{b}] \to \mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

then, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f* (*x*) is not continuous in [5, 9].

The differentiability of *f* in (5, 9) is checked as follows.

Let *n* be an integer such that *n* ∈ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of *f* at *x* = *n* are not equal, *f* is not differentiable at *x* = *n*

∴*f* is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function *f* (*x*) is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n-1-n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n-n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1,\ 2]$.

It can be proved as follows.

$$f(1) = 1^2 - 1 = 0,\ f(2) = 2^2 - 1 = 3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

$$f'(x) = 2x$$

$$\therefore f'(c) = 3$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$$

Question 1:

$\left(3x^2-9x+5\right)^9$

Answer

$\text{Let } y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8\cdot\left(6x-9\right)\\&=9\left(3x^2-9x+5\right)^8\cdot 3\left(2x-3\right)\\&=27\left(3x^2-9x+5\right)^8\left(2x-3\right)\end{aligned}$$

Question 2:

$\sin^3 x+\cos^6 x$

Answer

$\text{Let } y=\sin^3 x+\cos^6 x$

$$\begin{aligned}\therefore \frac{dy}{dx}&=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&=3\sin^2 x\cdot\frac{d}{dx}\left(\sin x\right)+6\cos^5 x\cdot\frac{d}{dx}\left(\cos x\right)\\&=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot\left(-\sin x\right)\\&=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Question 3:

$\left(5x\right)^{3\cos 2x}$

Answer

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y=3\cos 2x\log 5x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x\cdot\frac{d}{dx}(\cos 2x)+\cos 2x\cdot\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x)+\cos 2x\cdot\frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[-2\sin 2x\log 5x+\frac{\cos 2x}{x}\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

$$\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0\le x\le 1$

Answer

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$$

Answer

$$\text{Let } y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \qquad ...(1)$$

$$\text{Then, } \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$= \frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$= \frac{1+\cos x}{\sin x}$$

$$= \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Question 7:

$(\log x)^{\log x}$, $x > 1$

Answer

$$\text{Let } y = (\log x)^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Question 8:

$\cos(a\cos x + b\sin x)$, for some constant *a* and *b*.

Answer

$$\text{Let } y = \cos(a\cos x + b\sin x)$$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \cdot \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \cdot \left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x) \cdot \sin(a\cos x + b\sin x)$$

Question 9:

$$(\sin x - \cos x)^{(\sin x - \cos x)}, \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$$

Answer

$$\text{Let } y=(\sin x-\cos x)^{(\sin x-\cos x)}$$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$

$$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$$

$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$

$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

Question 10:

$x^x+x^a+a^x+a^a$, for some fixed $a>0$ and $x>0$

Answer

$$\text{Let } y=x^x+x^a+a^x+a^a$$

$$\text{Also, let } x^x=u,\ x^a=v,\ a^x=w,\text{ and } a^a=s$$

$$\therefore y=u+v+w+s$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx} \qquad \ldots(1)$$

$$u=x^x$$

$$\Rightarrow \log u=\log x^x$$

$$\Rightarrow \log u=x\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \cdot 1 + x \cdot \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x\left[\log x + 1\right] = x^x\left(1 + \log x\right) \qquad ...(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad ...(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a + 0$$

$$= x^x\left(1 + \log x\right) + ax^{a-1} + a^x \log a$$

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

$$\text{Let } y = x^{x^2-3}+(x-3)^{x^2}$$

$$\text{Also, let } u = x^{x^2-3} \text{ and } v=(x-3)^{x^2}$$

$$\therefore y = u+v$$

Differentiating both sides with respect to *x,* we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

$$u = x^{x^2-3}$$

$$\therefore \log u = \log\left(x^{x^2-3}\right)$$

$$\log u = \left(x^2-3\right)\log x$$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

$$v=(x-3)^{x^2}$$

$$\therefore \log v = \log(x-3)^{x^2}$$

$$\Rightarrow \log v = x^2\log(x-3)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

It is given that, $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore\frac{dx}{dt}=\frac{d}{dt}[10(t-\sin t)]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}[12(1-\cos t)]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot[0-(-\sin t)]=12\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1} x + \sin^{-1} \sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\sin^{-1} \sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x+y = -xy$

$\Rightarrow (1+x)y = -x$

$\Rightarrow y = \frac{-x}{(1+x)}$

Differentiating both sides with respect to *x*, we obtain

$y = \frac{-x}{(1+x)}$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$ is a constant independent of *a* and *b*.

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$$= -c, \text{ which is constant and is independent of } a \text{ and } b$$

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t \cdot \frac{d}{dx}(t) + t \cdot \frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t)\right\}\right]$$

$$= a\left[\cos t - \{\cos t - t\sin t\}\right] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$

$$= \sec^2 t \cdot \frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real *x*, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when *x* ≥ 0, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When *x* < 0, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for all positive integers n.

Answer

To prove: $P(n): \frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\therefore$P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $P(k): \frac{d}{dx}\left(x^k\right) = kx^{k-1}$

It has to be proved that P(k + 1) is also true.

$$\begin{aligned} \text{Consider } \frac{d}{dx}\left(x^{k+1}\right) &= \frac{d}{dx}\left(x \cdot x^k\right) \\ &= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}\left(x^k\right) \qquad \text{[By applying product rule]} \\ &= x^k \cdot 1 + x \cdot k \cdot x^{k-1} \\ &= x^k + kx^k \\ &= (k+1) \cdot x^k \\ &= (k+1) \cdot x^{(k+1)-1} \end{aligned}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\frac{d}{dx}(\sin A)+\sin A\cdot\frac{d}{dx}(\cos B)$$
$$+\sin B\cdot\frac{d}{dx}(\cos A)+\cos A\cdot\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\cdot\frac{d}{dx}(A+B)=\cos B\cdot\cos A\frac{dA}{dx}+\sin A(-\sin B)\frac{dB}{dx}$$
$$+\sin B(-\sin A)\cdot\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Question 22:

If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

If $y=e^{a\cos^{-1}x}, -1\leq x\leq 1$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Answer

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$$\log y = a\cos^{-1} x \log e$$

$$\log y = a\cos^{-1} x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.